ON BRAIDED AND RIBBON UNITARY FUSION CATEGORIES

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Abstract. We prove that every braiding over a unitary fusion category is unitary and every unitary braided fusion category admits a unique unitary ribbon structure.

1. Introduction

The notion of ribbon (or premodular) unitary fusion categories is important for their applications to low-dimensional topology and topological quantum computation (see [8] and [9]). They also appear naturally as invariants in subfactor theory and quantum groups theory.

During the AIM conference on "Classifying Fusion Categories" (March 2012) a list of problem was posted, see http://aimpl.org/fusioncat/. In this note we answer the second question of Problem 3.3, posted by Zhenghan Wang. We prove that every braiding over a unitary fusion category is unitary and every unitary braided fusion category admits a unique unitary ribbon structure. As a consequence, if the underlying fusion category of a modular category is unitary, we may freely choose a braiding to obtain a unitary braided fusion category and then there is exactly one choice of a ribbon structure that will make the associated Rational Conformal Field Theory unitary.

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2. Preliminaries

In this note, we will use basic theory of fusion categories and braided fusion categories. For further details on these topics, we refer the reader to [3]. In this section we recall some definitions and results on unitary fusion categories. Much of the material which appears here can be found in [8].

2.1. **Unitary fusion categories.** A C^***-category** $\mathcal{D}$ is a $\mathbb{C}$-linear abelian category with an involutive antilinear contravariant endofunctor $*$ which is the identity on objects, the hom-spaces $\mathrm{Hom}_{\mathcal{D}}(X,Y)$ are Hilbert spaces and the norms satisfy

$$||fg|| \leq ||f||\,||g||, \quad ||f^*f|| = ||f||^2,$$

for all $f \in \mathrm{Hom}_{\mathcal{D}}(X,Y), g \in \mathrm{Hom}_{\mathcal{D}}(Y,Z)$, where f^* denote the image of f under $*$.

Let X and Y be objects in a C^*-category. A morphism $u : X \to Y$ is **unitary** if $uu^* = \mathrm{id}_Y$ and $u^*u = \mathrm{id}_X$. A morphism $a : X \to X$ is **self-adjoint** if $a^* = a$.

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Remark 2.1. Every isomorphism in a C^*-category has a polar decomposition, *i.e.*, if $f : X \to Y$ is an isomorphism, then $f = ua$ where $a : X \to X$ is self-adjoint and $u : X \to Y$ is unitary, see [1, Proposition 8].

A **unitary fusion category** is a fusion category $\mathcal{C}$, where $\mathcal{C}$ is a C^*-category, the constraints are unitary and $(f \otimes g)^* = f^* \otimes g^*$, for every pair of morphisms f, g in $\mathcal{C}$.

Remark 2.2.

(1) A unitary fusion category is a fusion category with addition structure. Hence, a fusion category could have more than one unitary structure. All examples known to the author admit a unique unitary structure. Moreover, in [4, Theorem 5.20] it was proved that every weakly group-theoretical fusion category admits a unique unitary structure.
(2) If $\mathcal{C}$ is a unitary fusion category, we can find basis such that the F-matrices $(F_l^{ijk})_{n,m} = F_{l;n,m}^{i,j,k}$ are unitary, where $\{F_{l;n,m}^{i,j,k}\}$ are the $6j$-symbols (see [9] or [8] for the definition of $6j$-symbols). Conversely, if for a fusion category $\mathcal{C}$ it is possible to find basis such that the F-matrices $(F_l^{ijk})_{n,m} = F_{l;n,m}^{i,j,k}$ are unitary, then $\mathcal{C}$ is a unitary fusion category. See [10, Section 4].

3. Braiding and modular structures over unitary fusion categories are unitary

3.1. **The center of a unitary fusion category.** We shall recall the definition of the **center** $\mathcal{Z}(\mathcal{C})$ of a monoidal category $\mathcal{C}$, see [5, Chapter XIII]. The objects of $\mathcal{Z}(\mathcal{C})$ are pairs $(Y, c_{-,Y})$, where $Y \in \mathcal{C}$ and $c_{X,Y} : X \otimes Y \to Y \otimes X$ are isomorphisms natural in X satisfying $c_{X\otimes Y,Z} = (c_{X,Z} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes c_{Y,Z})$ and $c_{I,Y} = \mathrm{id}_Y$, for all $X, Y, Z \in \mathcal{C}$. A morphism $f : (X, c_{-,X}) \to (X, c_{-,X})$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that $(f \otimes \mathrm{id}_W)c_{W,X} = c_{W,Y}(\mathrm{id}_W \otimes f)$ for all $W \in \mathcal{C}$.

The center is a braided monoidal category with structure given as follows:

- the tensor product is $(Y, c_{-,Y}) \otimes (Z, c_{-,Z}) = (Y \otimes Z, c_{-,Y\otimes Z})$, where
$$c_{X,Y\otimes Z} = (\mathrm{id}_Y \otimes c_{X,Z})(c_{X,Y} \otimes \mathrm{id}_Z) : X \otimes Y \otimes Z \to Y \otimes Z \otimes X,$$
for all $X \in \mathcal{C}$,
- the identity element is $(I, c_{-,I})$, $c_{Z,I} = \mathrm{id}_Z$
- the braiding is given by the morphism $c_{X,Y}$.

If $\mathcal{C}$ is a unitary fusion category the **unitary center** $\mathcal{Z}^*(\mathcal{C})$ is defined as the full tensor subcategory of $\mathcal{Z}(\mathcal{C})$, where $(X, c_{-,X}) \in \mathcal{Z}^*(\mathcal{C})$ if and only if $c_{W,X} : W \otimes X \to X \otimes W$ is unitary for all $W \in \mathcal{C}$.

Proposition 3.1. *Let $\mathcal{C}$ be a unitary fusion category then $\mathcal{Z}^*(\mathcal{C}) = \mathcal{Z}(\mathcal{C})$.*

Proof. Let $(X, c_{-,X})$ be an object in $\mathcal{Z}(\mathcal{C})$. By [4, Proposition 5.24.] or [7, Theorem 6.4], the inclusion functor $\mathcal{Z}^*(\mathcal{C}) \subseteq \mathcal{Z}(\mathcal{C})$ is a tensor equivalence. Therefore, there is an object $(Y, c_{-,Y})$ in $\mathcal{Z}^*(\mathcal{C})$ and an isomorphism $f : (X, c_{-,X}) \to (Y, c_{-,Y})$ in $\mathcal{Z}(\mathcal{C})$. By Remark 2.1 there exists a unitary arrow $u : (X, c_{-,X}) \to (Y, c_{-,Y})$. Hence, for every $W \in \mathcal{C}$, $c_{W,X} = (u \otimes \mathrm{id}_W)^* \circ c_{W,Y} \circ (\mathrm{id}_W \otimes u)$, so $c_{W,X}$ is an unitary arrow and $(X, c_{-,X}) \in \mathcal{Z}^*(\mathcal{C})$. □

A braiding over a unitary fusion category $\mathcal{C}$ is called **unitary braiding** if the morphism $c_{X,Y}$ is unitary for any pair of objects $X, Y \in \mathcal{C}$.

Theorem 3.2. *Every braiding of a unitary fusion category is unitary.*

Proof. Let $\mathcal{C}$ be a unitary fusion category and let c be a braiding. It is easy to see that the braiding c defines an inclusion functor $\mathcal{C} \hookrightarrow \mathcal{Z}(\mathcal{C}), X \mapsto (X, c_{X,-})$. Proposition 3.1 implies that $c_{X,W}$ is unitary for every $W \in \mathcal{C}$. □

Remark 3.3.

(1) Theorem 3.2 implies that if the F-matrices $(F_l^{ijk})_{n,m} = F_{l;n,m}^{i,j,k}$ are unitary, then the R-matrices of the braiding are always unitarily diagonalizable.
(2) A Kac algebra $(H, m, \Delta, *)$ is a semisimple Hopf algebra such that $(H, *)$ is a C^*-algebra and the maps Δ and ε are C^*-algebra maps. Theorem 3.2 implies that every R-matrix in a Kac algebra is unitary in the sense that $R^* = R^{-1}$.

3.2. **Ribbon structures on unitary fusion categories.** If $\mathcal{C}$ is a fusion category, then for every $f \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$ the **transpose** of f, is defined by ${}^t f := (X^* \otimes \operatorname{ev}_Y)(X^* \otimes f \otimes Y^*)(\operatorname{coev}_X \otimes Y^*) \in \operatorname{Hom}_{\mathcal{C}}(Y^*, X^*)$.

A **twist** on a braided fusion category $\mathcal{C}$ is a natural automorphism of the identity functor $\theta \in \operatorname{Aut}(\operatorname{Id}_{\mathcal{C}})$, such that

$$\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) c_{Y,X} c_{X,Y}$$

for all $X, Y \in \mathcal{C}$. A twist is called a **ribbon structure** if ${}^t\theta_X = \theta_{X^*}$. A fusion category with a ribbon structure is called a **ribbon fusion category**. Each ribbon structure θ defines a **quantum dimension function** by $\dim_\theta(X) = \operatorname{ev}_X c_{X,X^*}(\theta_X \otimes \operatorname{id}_{X^*})\operatorname{coev}_X$.

We shall denote by $\operatorname{Aut}_{\otimes}(\operatorname{Id}_{\mathcal{C}})_{(+,-)}$ the abelian group of tensor automorphisms of the identity γ such that $\gamma_X = \pm \operatorname{id}_X$ for every simple object $X \in \mathcal{C}$.

Proposition 3.4. *Let $\mathcal{C}$ be a braided fusion category. If the set of ribbon structures is not empty, it is a torsor under* $\operatorname{Aut}_{\otimes}(Id_{\mathcal{C}})_{(+,-)}$.

Proof. Let θ and θ' be ribbon structures. It is easy to see that $\gamma := \theta^{-1} \circ \theta' : \operatorname{Id}_{\mathcal{C}} \to \operatorname{Id}_{\mathcal{C}}$ is a tensor automorphism of the identity. We may and shall assume that $\mathcal{C}$ is skeletal. Then, $X = X^{**}$ and for every simple object, we have $\theta_X = \theta(X)\operatorname{id}_X$, $\theta'_X = \theta(X)'\operatorname{id}_X$, $\gamma_X = \gamma(X)\operatorname{id}_X$ for some $\gamma(X), \theta(X), \theta(X)' \in \mathbb{C}^*$ and $\theta(X)' = \gamma(X)\theta(X)$. Since θ' is a ribbon structure, for every simple object $X \in \mathcal{C}$, $\dim_{\theta'}(X) = \dim_{\theta'}(X^*)$. On the other hand, $\dim_{\theta'}(X) = \gamma(X)\dim_\theta(X)$. Therefore $\gamma(X) = \gamma(X^*)$ and, since $\gamma(X^*) = \gamma(X)^{-1}$ we conclude that γ has order two.

Conversely, if γ is an automorphism of the identity such that $\gamma_X = \pm \operatorname{id}_X$ for every simple object, then, for every ribbon structure θ, the natural isomorphism $\theta' = \theta\gamma$ is a new ribbon structure. □

If $\mathcal{C}$ is a unitary fusion category a ribbon structure on $\mathcal{C}$ is called **unitary ribbon structure** if θ_X is unitary, $(\operatorname{coev}_X)^* = \operatorname{ev}_X \circ c_{X,X^*} \circ (\theta_X \otimes \operatorname{id}_{X^*})$ and $(\operatorname{ev}_X)^* = (\operatorname{id}_{X^*} \otimes \theta_X^{-1}) \circ c_{X^*,X}^{-1} \circ \operatorname{coev}_X$ for all $X \in \mathcal{C}$. A unitary fusion category with a unitary ribbon structure is called a **unitary ribbon fusion category**. In a unitary ribbon fusion category

$$\dim_\theta(X) = \operatorname{ev}_X \circ c_{X,X^*} \circ (\theta_X \otimes \operatorname{id}_{X^*}) \circ \operatorname{coev}_X = (\operatorname{coev}_X)^* \circ \operatorname{coev}_X,$$

therefore, the quantum dimension of every object is a positive number.

Theorem 3.5. *Every braided fusion category with a unitary structure admits a unique unitary ribbon structure.*

Proof. By [6, Proposition 2.4] every braided unitary fusion category admits a canonical unitary ribbon structure. Let θ_c the canonical ribbon structure associated to c. By Proposition 3.4, if θ' is another unitary ribbon structure, then there is $\gamma \in \operatorname{Aut}_{\otimes}(\mathrm{Id}_{\mathcal{C}})_{(+,-)}$ such that $\theta' = \theta_c\gamma$. If γ is not the identity there is a simple object $X \in \mathcal{C}$ such that $\gamma_X = -\mathrm{id}_X$, then $\dim_{\theta'}(X) = -\dim_{\theta_c}(X) < 0$, but the quantum dimension of every object of any unitary ribbon structure is positive. Therefore γ is the identity and θ_c is unique. $\square$

Remark 3.6. It follows from Theorem 3.5 that if a unitary braided fusion category is non-degenerate (see [2] for a definition), then it admits a unique unitary modular structure.

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